ON A THEOREM BY P. MALLIAVIN

BY

ÅKE PLEIJEL

ABSTRACT

P. Malliavin has deduced the asymptotic behaviour of a measure from the behaviour of its Stieltjes transform when $z \to \infty$ along a curve. His result is here proved in an elementary way.

0. Recently P. Malliavin [1] published a tauberian theorem for Stieltjes-transforms considered in the complex plane. His result has already been used by S. Agmon in a study of eigenvalues of differential operators which will appear in a forthcoming paper. I have observed that Malliavin's theorem is easily obtained by a method which I once used to study tauberian theorems [2]. Since I have understood that Malliavin's proof follows other lines I have considered it worth while to publish my way of proving his result.

1. **A basic formula.** Let

$$f(z) = \int_0^\infty (\lambda - z)^{-1} d\sigma(\lambda)$$

and let $L(Z)$ be an oriented curve in the complex plane from $\bar{Z}$ to $Z = X + iY$ not cutting the line of integration $0 \leqq x < \infty$. We assume $X > 0,\ Y > 0$.

By a change of the order of integration

$$I(Z) = \frac{1}{2\pi i} \int_{L(Z)} f(z) dz = \frac{1}{\pi} \int_0^\infty v(\lambda, Z) d\sigma(\lambda) \tag{1}$$

where v is the angle between the negative real direction and the direction from $(\lambda, 0)$ to Z.

We observe that if $f = f_1 + if_2$, f_1 and f_2 real, then

$$Yf_1(Z) = \int_0^\infty \tfrac{1}{2} \sin 2v \, d\sigma(\lambda),$$

$$Yf_2(Z) = \int_0^\infty \sin^2 v \, d\sigma(\lambda).$$

Formula (1) can be written

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(2) $$I(Z) - \frac{1}{\pi} Yf_1(Z) - \sigma(X) + \sigma(0) =$$

$$= \frac{1}{\pi}\int_0^X (v - \pi - \tfrac{1}{2}\sin 2v)\, d\sigma(\lambda) + \frac{1}{\pi}\int_X^\infty (v - \tfrac{1}{2}\sin 2v)\, d\sigma(\lambda).$$

Here

(3) $$\left| v - \pi - \tfrac{1}{2}\sin 2v \right| \leqq c\sin^2 v, \qquad \frac{\pi}{2} \leqq v \leqq \pi,$$

(4) $$\left| v - \tfrac{1}{2}\sin 2v \right| c\sin^2 v, \qquad 0 \leqq v \leqq \frac{\pi}{2},$$

where c is a constant. It is evident that $\sin^2 v$ can be replaced by $\sin^3 v$ which, however, does not improve the results in the considered cases. From (2), (3), (4) it follows, provided $d\sigma(\lambda) \geqq 0$, that

(5) $$\left| I(Z) - \frac{1}{\pi} Yf_1(Z) - \sigma(X) + \sigma(0) \right| \leqq \frac{c}{\pi} Yf_2(Z).$$

2. **Malliavin's theorem.** Malliavin considers a curve L which for $x \geqq 0$ is given by $y = \pm x^\gamma$, $0 \leqq \gamma < 1$, and assumes for f of section 1 that

(6) $$f(z) = a(-z)^\alpha + O(z^\beta)$$

when $z \to \infty$ along L. Here $-1 < \alpha < 0$ and $\beta < \alpha$. It is also supposed that $d\sigma(\lambda) \geqq 0$.

Evidently L may be modified near the origin so as to give a connected curve not cutting the positive real axis $x \geqq 0$. Since $\beta < \alpha$ it follows from (6) that

$$Yf(Z) = O(X^{\alpha+\gamma})$$

when $Z \to \infty$ along L.

We define $L(Z)$ as the part of the modified curve L for which $\operatorname{Re} z \leqq X$ when $Z = X + iY$. $O(\ )$ refers to the case when $Z \to \infty$ along L.

Write

(7) $$I(Z) = \frac{1}{2\pi i}\int_{L(Z)} (f(z) - a(-z)^\alpha)dz + \frac{a}{2\pi i}\int_{L(Z)} (-z)^\alpha dz.$$

To estimate the first integral of this formula we consider separately the cases $-1 < \beta < 0$, $\beta = -1$, $\beta < -1$. In the first case the integral is $O(X^{\beta+1})$, in the second it is $O(\log X)$. In the third case the integral converges to some constant A. The rate in which it approximates to A is $O(X^{\beta+1})$.

The second integral of (7) equals

$$aR^{\alpha+1}\frac{\sin(\alpha+1)(\pi-\psi)}{\pi(\alpha+1)},$$

where $R = |Z|$, $\psi = \arg Z$. This expression is written

$$aX^{\alpha+1}\frac{\sin\pi(\alpha+1)}{\pi(\alpha+1)} + O(X^{\alpha+\gamma}).$$

All taken together we find by the help of (5) that when $X \to \infty$

$$\sigma(X) - \sigma(0) = aX^{\alpha+1}\frac{\sin\pi(\alpha+1)}{\pi(\alpha+1)} + \\ + O(X^{\alpha+\gamma}) + O(X^{\beta+1}) + A,$$

where A is present only when $\beta < -1$ and the term $O(X^{\beta+1})$ should be replaced by $O(\log X)$ when $\beta = -1$. (The ε in Malliavin's remainder is not necessary.)

BIBLIOGRAPHY

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2. Pleijel, Åke, 1952, On a theorem of Carleman, *Matematisk Tidskrift*, B, 39–43.

MATEMATISKA INSTITUTIONEN
LUND, SWEDEN